

# Chiral symmetry and spectrum of Euclidean Dirac operator in QCD <sup>1</sup>

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March 1995

## Abstract

Some exact relations for the spectral density  $\rho(\lambda)$  of the Euclidean Dirac operator in  $QCD$  are derived. They follow directly from the chiral symmetry of the  $QCD$  lagrangian with massless quarks. New results are obtained both in thermodynamic limit when the Euclidean volume  $V$  is sent to infinity and also in the theory defined in finite volume where the spectrum is discrete and a nontrivial information on  $\rho(\lambda)$  in the region  $\lambda \sim 1/(|\langle \bar{q}q \rangle_0|V)$  (the characteristic level spacing) can be obtained. These exact results should be confronted with "experimental" numerical simulations on the lattices and in some particular models for  $QCD$  vacuum structure and may serve as a nontrivial test of the validity of these simulations.

## 1 Introduction

The notion of spectral density of the Euclidean Dirac operator in  $QCD$  has been brought into discussion some years ago in the pioneering paper by Banks and Casher [1]. They have got the famous formula

$$\rho(0) = -\frac{1}{\pi} \langle \bar{q}q \rangle_0 \quad (1.1)$$

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<sup>1</sup>Talk given at the International Conference on *Chiral Dynamics in Hadrons and Nuclei*, Seoul, February 1995.

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relating the infrared limit of the averaged spectral density with the quark condensate — the order parameter for the spontaneous chiral symmetry breaking.  $\rho(\lambda)$  is defined as

$$\rho(\lambda) = \frac{1}{V} \langle \omega(\lambda, A) \rangle_A \quad (1.2)$$

where

$$\omega(\lambda, A) = \sum_n \delta(\lambda - \lambda_n) \quad (1.3)$$

is the microscopic spectral density of the Dirac operator in a given background field  $A$  in a finite Euclidean volume  $V$ . The averaging is taken over all gluon fields with the standard Yang-Mills measure involving the determinant factor for  $N_f$  fermions with the common mass  $m$ . The combination  $V\rho(\lambda)d\lambda$  defines the average number of eigenvalues of Dirac operator in the interval  $(\lambda, \lambda + d\lambda)$ .

Let us recall how the Banks-Casher relation (1.1) (which holds in the thermodynamic limit  $V \rightarrow \infty$ , with the mass  $m$  being kept small but fixed) is derived. Treating the gauge field  $A_\mu^a(x)$  as an external field, the fermion Green's function is given by

$$S_A(x, y) = \langle q(x) \bar{q}(y) \rangle_A = \sum_n \frac{u_n(x) u_n^\dagger(y)}{m - i\lambda_n} \quad (1.4)$$

where  $u_n(x)$  and  $\lambda_n$  are eigenfunctions and eigenvalues of the Euclidean Dirac operator:

$$\not{D} u_n(x) = \lambda_n u_n(x) \quad (1.5)$$

Note that the spectrum of the theory enjoys the chiral symmetry:

$$\psi_n \rightarrow \gamma^5 \psi_n, \lambda_n \rightarrow -\lambda_n \quad (1.6)$$

And, except for zero modes, the eigenfunctions occur in pairs with opposite eigenvalues. Setting  $x = y$  and integrating over  $x$ , the representation (1.4) therefore implies

$$\frac{1}{V} \int_V dx \langle q(x) \bar{q}(x) \rangle_A = -\frac{2m}{V} \sum_{\lambda_n > 0} \frac{1}{m^2 + \lambda_n^2} \quad (1.7)$$

where the zero mode contributions have been dropped out (it is justified in the thermodynamic limit. See Ref.[2] for details). In the limit  $V \rightarrow \infty$ , the level spectrum becomes dense and we can trade the sum in the r.h.s. of Eq.(1.7) for the integral. We get

$$\langle \bar{q}q \rangle_0 = -2m \int_0^\infty d\lambda \frac{\rho(\lambda)}{m^2 + \lambda^2} \quad (1.8)$$

The integral (1.8) diverges at large  $\lambda$ . For  $\lambda \gg \Lambda_{QCD}$  the spectral density is not sensitive to gluon vacuum fluctuations and behaves in the same way as for free fermions:

$$\rho^{free}(\lambda) \sim \frac{N_c \lambda^3}{4\pi^2} \quad (1.9)$$

This perturbative quadratically divergent piece in  $\langle \bar{q}q \rangle_0$  is proportional to the quark mass and is related just to the fact that mass terms in the lagrangian break the chiral symmetry explicitly. To get the truly non-perturbative quark condensate which is the order parameter of the *spontaneous* chiral symmetry breaking, this perturbative divergent part should be subtracted. As a result, the quark condensate is related to the region of small  $\lambda$ :  $\lambda \sim m \ll \Lambda_{QCD}$ . It is not difficult to see that the finite mass-independent contribution appears if  $\rho(0) \neq 0$  and the relation (1.1) holds.

## 2 Thermodynamic limit: $\lambda \neq 0$ .

In this section, we derive a new formula describing the behavior of the spectral density  $\rho(\lambda)$  in the region  $m \ll \lambda \ll \Lambda_{QCD}$  in the thermodynamic limit  $V \rightarrow \infty$  [3]. To this end, let us consider the 2-point correlator

$$K^{ab} = \int d^4x \int d^4y \langle 0 | S^a(x) S^b(y) | 0 \rangle \quad (2.1)$$

with  $S^a = \bar{q} t^a q$  where  $t^a$  is the generator of the *flavour*  $SU(N_f)$  group. The correlator  $K^{ab}$  can be evaluated in the same way as  $\langle \bar{q}q \rangle_0$ . First, fix a particular gluon background. As  $Tr\{t^a\} = 0$ , only the connected part depicted in Fig.1a contributes, and one obtains

$$K_A^{ab} = - \int d^4x \int d^4y Tr\{t^a \mathcal{G}_A(x, y) t^b \mathcal{G}_A(y, x)\} \quad (2.2)$$

Substituting here the spectral decomposition for the Green's function (1.4) and taking into account the chiral symmetry of the spectrum (1.6), we get

$$K_A^{ab} = -\delta^{ab} \sum_{\lambda_n > 0} \frac{m^2 - \lambda_n^2}{(m^2 + \lambda_n^2)^2} \quad (2.3)$$

Rewriting the sum as an integral as in Eq.(1.8) and averaging over the gluon background, one gets

$$\frac{1}{V} K^{ab} = -\delta^{ab} \int_0^\infty \frac{\rho(\lambda)(m^2 - \lambda^2)}{(m^2 + \lambda^2)^2} d\lambda \quad (2.4)$$

On the other hand, the correlator (2.1) of the colorless scalar currents can be evaluated by inserting the complete set of physical states. For small  $m$ , i.e. close to the chiral limit, a distinguished position among the latter belongs to the Goldstone states which appear due to spontaneous breaking of the chiral symmetry of the QCD lagrangian. Low energy properties of Goldstones are fixed by chiral symmetry, and some exact calculations are possible. The chiral lagrangian has the form [4]

$$\mathcal{L} = F_\pi^2 \left[ \frac{1}{4} Tr\{(\partial_\mu U^\dagger)(\partial_\mu U)\} + B \text{Re} Tr\{\mathcal{M}U^\dagger\} \right] + \text{higher order terms} \quad (2.5)$$

where  $B = -\langle \bar{q}q \rangle_0 F_\pi^{-2}$ ,  $U = \exp\{2i\phi^a t^a / F_\pi\}$ , and  $\mathcal{M}$  is the quark mass matrix. If  $\mathcal{M} = \text{diag}(m, \dots, m)$ , the lagrangian (2.5) describes  $N_f^2 - 1$  (quasi-) Goldstone states with the common mass

$$M_\phi^2 = 2mB \quad (2.6)$$

Consider now the graph in Fig.1b with 2-goldstone intermediate state contributing to the correlator (2.4) (obviously, one-goldstone state does not contribute since  $\langle 0|S^a|\phi^b \rangle = 0$ ). To calculate it, we need to know the vertex  $\langle 0|S^a|\phi^b \phi^c \rangle$ . It can be easily found from the generating functional involving external scalar sources. The latter is obtained substituting in the effective lagrangian (2.5) the mass matrix  $\mathcal{M}$  by  $\mathcal{M} + u^a t^a$  where  $u^a$  is the source for the scalar current  $S^a$ . In this way, one gets

$$\langle 0|S^a|\phi^b \phi^c \rangle = B d^{abc} \quad (2.7)$$

It is very important that the vertex is nonzero only for three or more flavors. Now we can calculate the graph in Fig.1b absorbing its ultraviolet divergence into local counterterms contained in higher-order terms in the effective lagrangian (2.5) [4]. The result reads

$$\frac{1}{V} K^{ab} = -\frac{B^2(N_f^2 - 4)}{32\pi^2 N_f} \delta^{ab} \ln(M_\phi^2 / \mu_{had}^2) \quad (2.8)$$

For massless goldstones, the graph in Fig.1b does exhibit a logarithmic infrared singularity reflected in the factor  $\ln M_\phi^2$  on the r.h.s. of Eq.(2.8). The same circumstance makes our calculation self-consistent: since infrared singularity arises from the low momenta region, higher derivative terms in Eq.(2.5) can be neglected.

Now, let us compare Eq.(2.4) with Eq.(2.8). Note first of all that, in contrast to the integral in Eq.(1.8), the constant part  $\rho(0)$  does not contribute here at all:

$$\int_0^\infty \frac{\rho(0)(m^2 - \lambda^2)}{(m^2 + \lambda^2)^2} d\lambda = 0 \quad (2.9)$$

Thus, only the difference  $\rho(\lambda) - \rho(0)$  is relevant. It is easy to see that, in order to reproduce the singularity  $\propto \ln M_\phi^2 \propto \ln m$ , one should have

$$\rho(\lambda) - \rho(0) = C\lambda \quad (2.10)$$

at small  $\lambda$ . Comparing the coefficients at the logarithms  $\ln m$  and  $\ln M_\phi^2$  in Eq.(2.4) and Eq.(2.8), we arrive at the result

$$\rho(\lambda) = -\frac{1}{\pi} \langle \bar{q}q \rangle_0 + \frac{\langle \bar{q}q \rangle_0^2 (N_f^2 - 4)}{32\pi^2 N_f F_\pi^4} |\lambda| + o(\lambda) \quad (2.11)$$

The structure  $|\lambda|$  appears due to the relation  $\rho(-\lambda) = \rho(\lambda)$  being implied by the chiral symmetry (1.6). Note again that this non-analyticity does not appear at  $N_f = 2$ .

Recently, the preliminary, not yet published data for the spectral density  $\rho(\lambda)$  calculated on the lattice with two dynamical fermion flavours appeared [5]. The same quantity has been also evaluated in the instanton-antiinstanton liquid model for QCD vacuum configurations [6]. Results of these two numerical studies are completely different. Lattice calculation failed to reproduce the exact QCD relation (2.11) (they exhibit a large nonzero slope for  $\rho(\lambda)$  which should be absent for  $N_f = 2$ ). Probably, it is due to the fact that, in the region of lattice parameters used in [5], the thermodynamic limit was not reached yet and the finite volume effects were still important [7]. Obviously, further studies in this direction are highly desirable.

On the other hand, the calculations in the instanton model agree well with theoretical predictions. The results for the spectral density obtained in [6] for different number of light quark flavors are presented in Fig.2. Before comparing them with the theory, two remarks are in order.

1. Only non-perturbative instanton-driven part of spectral density has been determined in [6]. So, the perturbative effects which give the dominant contribution to the spectral density at large  $\lambda$  were not taken into account, and the comparison with theory makes sense only in the small  $\lambda$  region.
2. The abrupt falloff and vanishing of  $\rho(\lambda)$  at  $\lambda = 0$  as measured in [6] is the finite volume effect. The comparison should be done in the region of not yet too small  $\lambda$ .

From the graphs in Fig.2, one sees that, for  $N_f = 2$ , the spectral density is practically flat, while for  $N_f = 3$  it rises with a nonzero slope. This is exactly what the formula (2.11) requires.

The derivation of the result (2.11) assumed the spontaneous breaking of chiral symmetry and the existence of the Goldstone bosons. Thus, it refers only to the case  $N_f \geq 2$ . For  $N_f = 1$ , there is no theoretical result, and no comparison can be done. It is amusing, however, that, if trying to "continue analytically" the formula (2.11) down to the point  $N_f = 1$ , one would get a negative slope for  $\rho(\lambda)$  which agrees again with the instanton simulations.

Qualitatively, the different  $\lambda$  - dependence for different  $N_f$  is natural. Mean spectral density  $\rho(\lambda)$  is obtained after averaging of the microscopic spectral density (1.3) over gluon fields with the weight which involves also the quark determinant

$$[\det(i \mathcal{D})]^{N_f} \sim \left[ \prod_{\lambda_n > 0} \lambda_n^2 \right]^{N_f} \quad (2.12)$$

The larger  $N_f$  is, the more the region of small  $\lambda$  is suppressed. There are good reasons to expect that, at  $N_f = 0$ , with no suppression at all, quark condensate  $\equiv \rho(0)$  is infinite in the thermodynamic limit [8].<sup>3</sup>

### 3 Finite volume: the partition function.

In the rest of this talk, I derive and discuss some relations for the spectral density in the region of very small  $\lambda \sim 1/|\langle \bar{q}q \rangle|V$ . These relations refer not to physical QCD in the infinite volume, but to the theory defined in the finite Euclidean box. But, as our main goal is to discover islands of firm theoretical ground in the foreboding sea of numerical simulations, and the latter are done exclusively in finite volume, these intrinsically finite volume results can serve this purpose in exactly the same way as the result (2.11) derived in the infinite volume.

But before proceeding to the Dirac operator spectrum, we are in a position to study the more basic quantity — namely, the  $QCD$  partition function at finite volume. Specifically, we will be interested with the dependence of the partition function on the quark mass matrix  $\mathcal{M}$  and the vacuum angle  $\theta$ .

Consider first the theory with only one light quark flavor. It involves a gap in the spectrum ( $U(1)$  axial symmetry is broken not spontaneously but explicitly by anomaly, and no goldstones appear). Thus, the extensive property for the partition function holds:

$$Z \sim \exp\{-\epsilon_{vac}(m, \theta)V\} \quad (3.1)$$

If  $L \gg \Lambda_{QCD}^{-1}$ , the finite volume effects in  $\epsilon_{vac}$  are exponentially small [9]. Ward identities dictate that  $\epsilon_{vac}$  can depend on  $m$  and  $\theta$  not in an arbitrary way, but only as a function of a particular combination  $me^{i\theta}$ . Expanding  $\epsilon_{vac}(me^{i\theta})$  in Taylor series (that makes sense as long as  $m \ll \Lambda_{QCD}$ ) and bearing in mind that  $\epsilon_{vac}$  is real, we get

$$Z \sim e^{\Sigma V m \cos \theta + O(m^2)} \quad (3.2)$$

The parameter  $\Sigma$  is nothing else but the quark condensate (up to a sign). Really,

$$\langle \bar{q}q \rangle_0 = -\frac{d}{dm} \ln Z = -\Sigma \quad (3.3)$$

Consider now the same problem but for several quark flavors. Then spontaneous breaking of chiral symmetry occurs, goldstones appear, there is no gap in the spectrum

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<sup>3</sup>The instanton calculations of Ref.[6] do not exhibit the infinite quark condensate for the quenched theory, but only a much more dramatic rise of the spectral density in the low  $\lambda$  region. But, probably, the finite value for  $\rho(0)$  is, again, a finite-volume effect.

, and the extensive property (3.1) does not hold anymore. More exactly, if quark masses are non-zero, the property (3.1) still holds when the length of the box  $L$  is much larger than the Compton wavelength of Goldstone particles  $\sim 1/m_\phi \sim 1/\sqrt{m\Lambda_{QCD}}$ . We will be interested, however, with the intermediate region

$$\Lambda_{QCD}^{-1} \ll L \ll \frac{1}{\sqrt{m\Lambda_{QCD}}} \quad (3.4)$$

where the effects due to goldstones on the volume and mass dependence of the partition function are crucial.

Fortunately, the goldstone properties *at low energies* (and, in the region  $L \gg \Lambda_{QCD}^{-1}$ , only the low-energy properties of the Goldstone fields are relevant) are well known. They are described by the effective lagrangian (2.5) where, at nonzero  $\theta$ ,  $\mathcal{M}$  should be substituted by  $\mathcal{M}e^{i\theta/N_f}$  (again, the occurrence of this particular combination is dictated by Ward identities). The partition function is given by the functional integral

$$Z \sim \int [dU] \exp \left\{ -F_\pi^2 \int_V d^4x \left[ \frac{1}{4} \text{Tr} \{ (\partial_\mu U^\dagger) (\partial_\mu U) \} + B \text{Re Tr} \{ \mathcal{M} e^{i\theta/N_f} U^\dagger \} \right] \right\} \quad (3.5)$$

The crucial observation is that, as long as  $L \ll \frac{1}{m_\phi} \sim \frac{1}{\sqrt{m\Lambda_{QCD}}}$ , only zero Fourier harmonics  $U_0$  of the field  $U(x)$  is relevant, and the contribution of the higher harmonics in the integral (3.5) is suppressed. Then the functional integral is transformed into the finite-dimensional integral

$$Z \sim \int_{SU(N_f)} d\mu(U_0) \exp \{ V \Sigma \text{Re Tr} \{ \mathcal{M} e^{i\theta/N_f} U_0^\dagger \} \} \quad (3.6)$$

The integral extends over the group  $SU(N_f)$ , and  $d\mu(U_0)$  is the corresponding Haar measure. For  $N_f = 2$ , the integral (3.6) can be done explicitly. For  $N_f \geq 3$ , it is not possible to do analytically in the general case. Sometimes (for the case  $\mathcal{M} = m\hat{1}$ ), certain beautiful results can be obtained [2], but, to derive the *sum rules* for the small eigenvalues of the Dirac operator we are after in this talk, the closed analytic expression for  $Z(V, \mathcal{M}, \theta)$  is not actually required.

The partition function (3.6) is a periodic function of  $\theta$  with the period  $2\pi$ . Thus, it can be presented as a Fourier series

$$Z(\theta) = \sum_{\nu=-\infty}^{\infty} Z_\nu e^{i\nu\theta} \quad (3.7)$$

so that

$$Z_\nu = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z(\theta) \sim \int_{U(N_f)} d\mu(\tilde{U}) (\det \tilde{U})^\nu \exp \{ V \Sigma \text{Re Tr} \{ \mathcal{M} \tilde{U}^\dagger \} \} \quad (3.8)$$

with  $\tilde{U} = U_0 e^{-i\theta/N_f}$ .

## 4 Finite volume: sum rules.

The main idea is basically the same as in Sect.2 — to compare the expression (3.8) derived in the chiral theory with the quark-gluon representation of the same quantity.  $Z_\nu$  may be written as a functional integral over the quark and gluon fields restricted on the topological class with a given winding number

$$\nu = \frac{1}{32\pi^2} \int d^4x G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a \quad (4.1)$$

The integral over the Fermi fields produces the determinant of the Dirac operator, and we get for  $\nu > 0$

$$Z_\nu = \int [dA] e^{-\frac{1}{4} \int d^4x G_{\mu\nu}^a G_{\mu\nu}^a} (det_f \mathcal{M})^\nu \prod_{\lambda_n > 0} det_f(\lambda_n^2 + \mathcal{M}\mathcal{M}^\dagger) \quad (4.2)$$

where the factor  $(det_f \mathcal{M})^\nu$  arises due to the fermion zero modes  $\lambda_n = 0$  which appear on a topologically non-trivial background due to the index theorem. (If  $\nu$  is negative, the factor  $(det_f \mathcal{M})^\nu$  is to be replaced by  $(det_f \mathcal{M}^\dagger)^{-\nu}$ ).

Let us expand now Eqs.(3.8) and (4.2) in quark mass and compare the coefficients of  $(det_f \mathcal{M})^\nu \text{Tr}\{\mathcal{M}\mathcal{M}^\dagger\}$ . On the chiral side of the equality, we get some group integral which can be done explicitly. On the quark and gluon side, we get the expression involving  $\langle \sum_{\lambda_n > 0} 1/\lambda_n^2 \rangle_\nu$  where the average is defined as

$$\langle f \rangle_\nu = \frac{\int [dA] f e^{-\frac{1}{4} \int d^4x G_{\mu\nu}^a G_{\mu\nu}^a} (\prod_{\lambda_n > 0} \lambda_n^2)^{N_f}}{\int [dA] e^{-\frac{1}{4} \int d^4x G_{\mu\nu}^a G_{\mu\nu}^a} (\prod_{\lambda_n > 0} \lambda_n^2)^{N_f}} \quad (4.3)$$

where the path integral is done over all gauge fields with the topological charge  $\nu$ . Skipping the technical details, we present the final result for the simplest sum rule thus derived:

$$\left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{V^2 \Sigma^2}{4k} \quad (4.4)$$

where  $k = |\nu| + N_f$ .

Expanding the expressions (3.8) and (4.2) further in quark mass and comparing the coefficients of the two group invariants  $(det_f \mathcal{M})^\nu (\text{Tr}\{\mathcal{M}\mathcal{M}^\dagger\})^2$  and  $(det_f \mathcal{M})^\nu \text{Tr}\{\mathcal{M}\mathcal{M}^\dagger \mathcal{M}\mathcal{M}^\dagger\}$ , we get two different sum rules for the inverse fourth powers of eigenvalues

$$\left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^4} \right\rangle_\nu = \frac{V^4 \Sigma^4}{16k(k^2 - 1)}; \quad \left\langle \left( \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right)^2 \right\rangle_\nu = \frac{V^4 \Sigma^4}{16(k^2 - 1)} \quad (4.5)$$

(Two different sum rules (4.5) are obtained when  $N_f \geq 2$ . If  $N_f = 1$ , there is no difference between  $(\text{Tr}\{\mathcal{M}\mathcal{M}^\dagger\})^2$  and  $\text{Tr}\{\mathcal{M}\mathcal{M}^\dagger \mathcal{M}\mathcal{M}^\dagger\}$  and only one sum rule



involving inverse fourth powers of  $\lambda_n$  for the difference of the two sums entering Eq.(4.5) can be derived). Further expansion in  $\mathcal{M}$  provides the sum rules with inverse sixth powers of eigenvalues etc.

Note that the sums (4.4) , (4.5) are saturated by few first eigenvalues. Therefore the sum rules provide the information on the very bottom of the finite-volume spectrum.

The theoretical results (4.4) , (4.5) have not yet been confronted with lattice experiments. However, the numerical calculations in the instanton model in the case  $\nu = 0$  with different number of quark flavors are available. They are in a good agreement with the theory (see Fig.3 and the discussion thereafter).

## 5 Stochastic matrices

The sum rules (4.4) , (4.5) are exact theoretical results. They follow just from the form of  $QCD$  lagrangian and the assumption (which is the experimental fact in  $QCD$ ) that the spontaneous breaking of chiral symmetry occurs. It is interesting, however, that much richer information on the spectrum of Dirac operator in the region  $\lambda \sim 1/(\Sigma V)$  can be obtained if invoking a certain *additional assumption* on the properties of the functional integral in  $QCD$  [10].

Consider the Dirac operator in a particular gauge field background with the topological charge  $\nu$ . For simplicity, let us assume that the mass matrix is diagonal and real. Let us choose a particular basis  $\mathcal{E} = \{\psi_n(x)\}$  and present the Dirac operator  $-i \not{D} + m_f$  as a matrix in this basis. The particular choice of the basis is not important for us, but a reader familiar with the model of instanton-antiinstanton liquid may think of  $\psi_n(x)$  as of zero-mode solutions for *individual* instantons and antiinstantons. What *is* important is that the basis  $\mathcal{E}$  is chosen in such a way that all states  $\psi_n(x)$  have a definite (left or right) chirality. In that case, the number of left-handed states  $n_L$  and the number of right-handed states  $n_R$  in the basis should be related:

$$n_L - n_R = \nu \quad (5.1)$$

Thereby, the existence of exactly  $\nu$  left-handed ( $\nu > 0$ ) or  $-\nu$  right-handed ( $\nu < 0$ ) zero eigenvalues of the massless Dirac operator  $-i \not{D}$  is assured. In this basis, the full Dirac operator  $-i \not{D} + m_f$  can be presented as a matrix

$$-i \not{D} + m_f = \begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix} \quad (5.2)$$

where  $T$  is a general rectangular complex matrix  $n_L \times n_R$ . We assume  $n_L, n_R$  to be very large and will eventually send them to infinity. In this limit, the basis  $\mathcal{E}$  spans the whole Hilbert space and the representation (5.2) is just exact.

The partition function involves the Dirac determinant and the integral over all gluon configurations. If the basis  $\mathcal{E}$  is fixed (the same for all gluon field configurations), the path integral over the latter can be traded for an integral over the matrices  $T$ . Thus, we have

$$Z_\nu = \int \mathcal{D}T P(T) \prod_f^{N_f} \det \begin{pmatrix} m_f & iT \\ iT^\dagger & m_f \end{pmatrix} \quad (5.3)$$

, the condition (5.1) for the dimensions of the matrix  $T$  being assumed.

And now comes the crucial assumption. Let us assume that the measure  $P(T)$  has the simple form

$$P(T) = \exp \left\{ -\frac{A}{n} \text{Tr} \{ TT^\dagger \} + o(TT^\dagger) \right\} \quad (5.4)$$

The constant  $A$  will at the end determine the spectral density  $\rho(0)$  in the thermodynamic limit.

The form (5.4) of the weight is rather natural. It respects chiral symmetry, analytic in  $T$ , and is conceptually similar to the Gibbs statistical distribution. However, a rigorous *proof* that  $P(T)$  *should* have this form is absent by now. Therefore, the results for the microscopic spectral density  $\rho(\lambda \sim 1/\Sigma V)$  derived in ([10]) which follow from the representation (5.3) for the partition function and the heuristic assumption (5.4) have not the same status as the sum rules (4.4), (4.5) which are the *theorems* of *QCD*.

Let us explain (very sketchy) how the results for the microscopic spectral density have been derived. For simplicity, assume that  $\nu = 0$  so that  $T$  is a square matrix:  $n_L = n_R \equiv n$ .  $P(T)$  and the Dirac determinant depend only on the eigenvalues  $\lambda_n$  of the matrix  $T$ . Therefore, it is convenient to write the integral over  $\mathcal{D}T$  as the integral over eigenvalues and do the integral over remaining angular variables on which the integrand does not depend. We have <sup>4</sup>

$$\mathcal{D}T = C \prod_{k < l} (\lambda_k^2 - \lambda_l^2)^2 \prod_k \lambda_k d\lambda_k \quad (5.5)$$

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<sup>4</sup>This transformation and the whole stochastic matrix technique is well known for nuclear physicists and solid state physicists who are concerned with the problem of quantum dynamic for stochastic hamiltonia. The particular problem studied here coincides with the known problem of "Gaussian unitary ensemble". The vanishing of the Jacobian at  $\lambda_k = \lambda_l$  reflects the notorious repulsion of the levels. For an extensive review of the relevant mathematics see [11].

Thus, joint eigenvalue distribution is just

$$\rho(\lambda_1, \dots, \lambda_n) \sim \prod_{k < l} (\lambda_k^2 - \lambda_l^2)^2 \prod_k \lambda_k \exp\left\{-\frac{A}{n} \sum_{k=1}^n \lambda_k^2\right\} \prod_f (\lambda_k^2 + m_f^2) \quad (5.6)$$

The microscopic spectral density is obtained after integration of (5.6) over all eigenvalues but one. After some work, one gets the answer in the limit  $n \rightarrow \infty$ . In [10] the result was obtained in the particular case  $\nu = 0$ . Being expressed via physical variables  $V$  (the volume of the system) and  $\Sigma$  (the quark condensate),  $A \equiv (\Sigma V)^2/4$ , it has the form

$$\rho(\lambda) = \frac{\Sigma^2 V \lambda}{2} \left[ J_{N_f}^2(\Sigma V \lambda) - J_{N_f+1}(\Sigma V \lambda) J_{N_f-1}(\Sigma V \lambda) \right] \quad (5.7)$$

where  $J_\mu(x)$  are the Bessel functions. The result (5.7) has been obtained in [10] under the assumption  $\nu = 0$ . In [12] it has been generalized to the case  $\nu \neq 0$ . [the only change is that  $N_f$  should be substituted by  $N_f + |\nu|$  as in Eqs. (4.4, 4.5)].

At very small  $\lambda \ll 1/V\Sigma$ , the spectral density is suppressed  $\rho(\lambda) \sim \lambda^{2N_f+1}$  (the suppression is due to the determinant factor (2.12) which punishes small eigenvalues). But then it rises and after several oscillations with decreasing amplitude levels off at a constant  $\rho(0) = \Sigma/\pi$  as it should in the thermodynamic limit  $V \rightarrow \infty$ . The function (5.7) with  $N_f = 0, 1, 2$  together with the results of numerical instanton calculations are plotted in Fig.3 borrowed from Ref.[6]. Likewise, one can integrate Eq.(5.6) over all eigenvalues but two and derive the expression for the correlation function  $\rho(\lambda_1, \lambda_2)$  etc.

The sum rules derived in the previous section can be expressed as the integrals of  $\rho(\lambda), \rho(\lambda_1, \lambda_2)$  etc. with a proper weight. For example, the simplest sum rule is

$$\left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_{\nu=0} = \int_0^\infty \frac{\rho(\lambda) d\lambda}{\lambda^2} = \frac{V^2 \Sigma^2}{4N_f} \quad (5.8)$$

which coincides with (4.4). Thus, the excellent agreement of the model instanton calculations with the stochastic matrix model results displayed in Fig.3 implies also the agreement of the former with the exact theoretical results (4.4), (4.5) etc.

## 6 Exotic theories.

The same program as in  $QCD$  can be carried out in other gauge theories with non-standard fermion content. The results derived earlier depended on the assumption of the standard pattern of chiral symmetry breaking

$$SU_L(N_f) \otimes SU_R(N_f) \rightarrow SU_V(N_f) \quad (6.1)$$

so that  $N_f^2 - 1$  Goldstone bosons living on the coset, which is also  $SU(N_f)$ , appear. However, the breaking according to this scheme only occurs for fermions belonging to the complex (e.g. fundamental) representation of the gauge group. The  $SU(N_c)$  group involves truly complex representations when  $N_c \geq 3$ . For  $N_c = 2$ , the fundamental representation (and also other representations with half-integer isospin) is pseudoreal: quarks and antiquarks transform in the same way under the action of the gauge group and the pattern of chiral symmetry breaking is different leading to different sum rules. A third pattern of chiral symmetry breaking is for fermions in the real (e.g. adjoint) representation leading to yet another class of sum rules.

Consider first the case of pseudoreal fermions. It may have a considerable practical importance as the numerical calculations with the  $SU(2)$  gauge group are simpler than with  $SU(3)$  group, and it may be easier to confront the sum rules with lattice simulations. The true chiral symmetry group of the  $QCD$ -like lagrangian but with  $SU(2)$  gauge group is  $SU(2N_f)$  rather than just  $SU_L(N_f) \otimes SU(N_R) \otimes U_V(1)$  (it involves also the transformations that mix quarks with antiquarks). The pattern of spontaneous symmetry breaking due to formation of quark condensate is [13]

$$SU(2N_f) \rightarrow Sp(2N_f) \quad (6.2)$$

The breaking (6.2) leads to the appearance of

$$(2N_f)^2 - 1 - (2N_f^2 + N_f) = 2N_f^2 - N_f - 1$$

Goldstone bosons which are parametrized by the coset  $SU(2N_f)/Sp(2N_f)$ . For  $N_f = 2$ , we have 5 instead of the usual 3 Goldstone bosons. Let us now allow for small non-zero quark masses (which break the chiral symmetry explicitly and give non-zero masses to Goldstone particles) and study the dependence of the partition function in the sector with a given topological charge  $\nu$  on the quark mass matrix  $\mathcal{M}$  and the volume  $V$ . The analog of the result (3.8) for the pseudoreal case which is valid in the region (3.4) is the following (see [14]) for more details)

$$Z_\nu \sim \int_{U(2N_f)} d\mu(U) (\det U)^{-\nu} \exp \left\{ \frac{V\Sigma}{2} \text{ReTr} \{ \mathcal{M} U I U^T \} \right\} \quad (6.3)$$

where

$$I = \begin{pmatrix} 0 & \hat{1} \\ -\hat{1} & 0 \end{pmatrix} \quad (6.4)$$

is the symplectic  $2N_f \otimes 2N_f$  matrix. Formally, the representation (6.3) involves the integral over  $U(2N_f)$ , but actually it is the integral over the coset  $SU(2N_f)/Sp(2N_f)$  involving less number of parameters (*and* the integral over the  $U(1)$  part of  $U(2N_f)$

which is nothing else but a vacuum angle  $\theta$ ) — multiplying  $U$  by a symplectic matrix  $\in Sp(2N_f)$  leaves the integrand invariant.

Expanding (6.3) over the group invariants involving the quark matrix  $\mathcal{M}$  and expanding also the quark-gluon representation of the same partition function, the sum rules for the inverse powers of the eigenvalues can be derived. The simplest sum rule is

$$\left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{V^2 \Sigma^2}{4(|\nu| + 2N_f - 1)} \quad (6.5)$$

Consider now the third non-trivial class of theories where the fermions belong to the real representation of the gauge group. The supersymmetric Yang-Mills theories involving Majorana fermions in the adjoint color representation belong to this class. We will not assume, however, that the theory is supersymmetric and consider a Yang-Mills theory coupled to  $N_f$  different Majorana adjoint fermion fields. Also, we will assume that the fermion condensate is formed and the chiral symmetry which the lagrangian enjoys for  $N_f \geq 2$  is spontaneously broken <sup>5</sup>.

The chiral symmetry group of such a theory is  $SU(N_f)$ . Formation of the condensate breaks it down to  $SO(N_f)$ , and

$$N_f^2 - 1 - \frac{N_f(N_f - 1)}{2} = \frac{N_f(N_f + 1)}{2} - 1$$

Goldstone bosons appear. Again, the partition function  $Z_\nu$  is given by the integral over the coset  $SU(N_f)/SO(N_f)$  which has the form

$$Z_\nu \sim \int_{U(N_f)} d\mu(U) (\det U)^{-2\nu N_c} \exp \left\{ V \Sigma \operatorname{Re} \operatorname{Tr} \{ \mathcal{M} U U^T \} \right\} \quad (6.6)$$

The essential difference with the case of fundamental fermions is that the admissible values for the topological charge  $\nu$  are not integer but integer multiples of  $1/N_c$ . That means, in particular, that the function  $Z(\mathcal{M}, \theta)$  given by the Fourier sum (3.7) is a periodic function of  $\theta$  with the period  $2\pi N_c$  (not  $2\pi$  as before). See Ref.[2] for a detailed discussion of this important issue.

To derive the sum rules, we have to expand (6.6) over  $\mathcal{M}$  and to compare it with the expansion of the quark-gluon path integral for the partition function. And here we meet a known problem characteristic for theories with Majorana particles. The matter is that the reality condition for the Fermi fields can be fulfilled in Minkowski space but not in Euclidean space - it immediately leads to a contradiction [16]

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<sup>5</sup> This dynamic assumption is not so innocent. For example, in  $N = 2$  supersymmetric Yang-Mills theory it does not happen [15]. But this theory involves also massless adjoint scalar fields with particular interaction vertices. It is quite conceivable that the non-supersymmetric theory with only gauge fields and two Majorana adjoint flavors *involves* chiral symmetry breaking. All the results are derived under this assumption.

However, the Euclidean path integral for the partition function in a theory with Majorana fermions still can be defined by analytic continuation of the Minkowski space path integral. In the Minkowski space, the path integral over fermion fields is equal to the *square root* of the corresponding Dirac determinant. The latter can be continued to Euclidean space without problems, and the square root is also taken easily here due to a notable fact that the spectrum of the Euclidean Dirac operator for the adjoint fermions is doubly degenerate: for any eigenfunction  $u_n(x)$  with an eigenvalue  $\lambda_n$ , the function  $C^{-1}u_n^*(x)$  ( $C$  is the charge conjugation matrix) is linearly independent from  $u_n(x)$  and has exactly the same eigenvalue [2]. Thus, square root of the Dirac determinant is just

$$[\det(-i \not{D} + \tilde{\mathcal{M}})]^{1/2} = (\det_f \mathcal{M})^{\nu N_c} \prod'_{\lambda_n > 0} \det_f(\lambda_n^2 + \mathcal{M} \mathcal{M}^\dagger) \quad (6.7)$$

where  $\tilde{\mathcal{M}} = \frac{1}{2}(1 - \gamma^5)\mathcal{M} + \frac{1}{2}(1 + \gamma^5)\mathcal{M}^\dagger$ , the product counts only one eigenvalue of each degenerate pair, and the factor  $(\det_f \mathcal{M})^{\nu N_c}$  reflects the presence of  $\nu N_c$  pairs of zero eigenmodes of adjoint Dirac operator in the gauge field background with the topological charge  $\nu$ <sup>6</sup>.

The final form of the simplest sum rule for the eigenvalues of the *Dirac* adjoint determinant is

$$\left\langle \sum'_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{V^2 \Sigma^2}{4(|\nu| N_c + (N_f + 1)/2)} \quad (6.8)$$

where, again, only one eigenvalue of each degenerate pair is taken into account in the sum.

The sum rules (4.4), (6.5), and (6.8) can be written universally as

$$\left\langle \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{V^2 \Sigma^2}{4\{|\nu| + [\dim(\text{coset}) + 1]/N_f\}} \quad (6.9)$$

with the rescaling  $\nu \rightarrow \nu N_c$  and counting in the sum only one eigenvalue of each degenerate pair in the adjoint case.

The sum rules (6.5), and (6.8) and their analogs with higher inverse powers of  $\lambda_n$  can be derived also in the stochastic matrix technique, and not only the sum rules, but also the expressions for the microscopic spectral density  $\rho(\lambda)$ , correlators  $\rho(\lambda_1, \lambda_2)$  etc. [18]

Mathematically, 3 classes of theories discussed (with complex fermions, pseudoreal fermions and real fermions) are described in terms of 3 classic stochastic en-

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<sup>6</sup>The nice result (6.7) is specific for the theories with *adjoint* Majorana fermions. To define in the Euclidean space a theory like the standard model involving chiral fermions in the fundamental representation is more difficult [17].

sembles: Gaussian unitary, Gaussian orthogonal, and Gaussian symplectic. It is noteworthy that, though the sum rules in all 3 cases are the same for  $N_f = 1$  [where no spontaneous breaking of chiral symmetry occurs, no Goldstone bosons appear, and the partition function has the extensive form (3.1)] , the microscopic spectral densities and the correlators are not. That elucidates again the fact that , when deriving microscopic spectral densities in the stochastic matrix technique, an extra assumption (5.4) beyond using just the symmetry properties of the theory is necessary to adopt.

## 7 Acknowledgements.

It is a pleasure for me to thank the organizers of the Seoul meeting for kind hospitality. I am indebted to J. Verbaarschot for valuable discussions and for kindly sending me the postscript files of Fig.2 and Fig.3. This work was supported in part by Schweizerischer Nationalfonds and the INTAS grant 93-0283.

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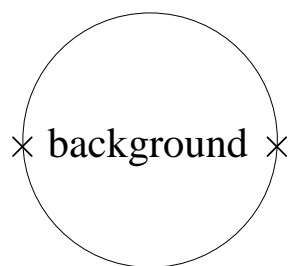
## Figure captions.

**Fig. 1.** 2-point correlator. (a) Quark representation. The loop is evaluated in a gauge-field background to be averaged over afterwards. (b) Quasi-Goldstone contribution.

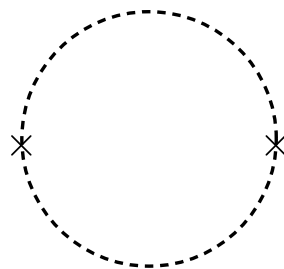
**Fig. 2.** Instanton calculations for the spectral density  $n(\lambda) = \rho(\lambda)$  for different  $N_f$ .  $\lambda$  is measured in units of  $\Lambda_{QCD}$ , and the area below the curve is normalized to 1.

**Fig. 3.** Microscopic spectral density as a function of  $z = \Sigma V \lambda / \pi$ . The normalization  $\rho(0) = 1$  is chosen. Dashed lines correspond to the theoretical result (5.7). Full lines present the results of numerical calculations in the instanton liquid model.





a)



b)

